

RATIONALITY OF ADMISSIBLE AFFINE VERTEX ALGEBRAS IN THE CATEGORY \mathcal{O}

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ABSTRACT. We study the vertex algebras associated with modular invariant representations of affine Kac-Moody algebras at fractional levels, whose simple highest weight modules are classified by Joseph's characteristic varieties. We show that an irreducible highest weight representation of a non-twisted affine Kac-Moody algebra at an admissible level k is a module over the associated simple affine vertex algebra if and only if it is an admissible representation whose integral root system is isomorphic to that of the vertex algebra itself. This in particular proves the conjecture of Adamović and Milas [AM] on the rationality of admissible affine vertex algebras in the category \mathcal{O} .

1. INTRODUCTION

Let \mathfrak{g} be a complex simple Lie algebra, $\widehat{\mathfrak{g}}$ the non-twisted affine Kac-Moody algebra associated with \mathfrak{g} , $V^k(\mathfrak{g})$ the universal affine vertex algebra associated with \mathfrak{g} at a non-critical level k , $L(k\Lambda_0)$ the unique simple quotient of $V^k(\mathfrak{g})$. The simple affine vertex algebra $L(k\Lambda_0)$ is called *admissible* if it is isomorphic to an admissible representation [KW1] as a $\widehat{\mathfrak{g}}$ -module. The purpose of this article is to classify simple modules over admissible affine vertex algebras.

By a well-known result of Zhu [Zhu], there is a one-to-one correspondence between positively graded simple modules over a graded vertex algebra V and simple $A(V)$ -modules, where $A(V)$ is *Zhu's algebra* of V . In the case that V is an affine vertex algebra $L(k\Lambda_0)$, we have

$$(1) \quad A(L(k\Lambda_0)) \cong U(\mathfrak{g})/J_k$$

for some two sided-ideal J_k of the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . Since a simple $U(\mathfrak{g})$ -module V is an $A(L(k\Lambda_0))$ -module if and only if the annihilating ideal of V in $U(\mathfrak{g})$ contains J_k , our problem is to classify the primitive ideals of $U(\mathfrak{g})$ containing J_k . Because any primitive ideal of $U(\mathfrak{g})$ is the annihilating ideal of a highest weight representation of \mathfrak{g} [Duf], it suffices to classify simple highest weight representations of $A(L(k\Lambda_0))$, or equivalently, to classify simple $L(k\Lambda_0)$ -modules in the category \mathcal{O} of $\widehat{\mathfrak{g}}$.

Let $L(\lambda)$ be the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight λ .

Main Theorem. *Let k be an admissible number¹, λ a weight of $\widehat{\mathfrak{g}}$ of level k . Then $L(\lambda)$ is a module over $L(k\Lambda_0)$ if and only if it is an admissible representation whose*

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¹A complex number k is called admissible if $L(k\Lambda_0)$ is an admissible representation, cf. Proposition 3.1.

integral root system is isomorphic to that of $L(k\Lambda_0)$. In particular the conjecture of Adamović and Milas [AM, Conjecture 3.5.7] holds, that is, any $L(k\Lambda_0)$ -module in the category \mathcal{O} of $\widehat{\mathfrak{g}}$ is completely reducible.

Main Theorem has been proved in some special cases: for type $C_\ell^{(1)}$ admissible half integer levels by Adamović [Ada]; for $\widehat{\mathfrak{sl}}_2$ by Adamović and Milas [AM], Dong, Li and Mason [DLM] and Feigin and Malikov [FeM]; for type $A_\ell^{(1)}, B_\ell^{(1)}$ admissible half integer levels by Perše [P1, P2]; for type $G_2^{(1)}$ admissible one-third integer levels by Axtell and Lee [AL] and Axtell [Axt]. These works are based on the explicit computation of the singular vector of $V^k(\mathfrak{g})$ which generates the maximal ideal. Our method in this article is different.

Let us explain the outline of the proof of Main Theorem briefly. By (1) simple highest weight $L(k\Lambda_0)$ -modules are classified by Joseph's *characteristic variety* [Jos] $\mathcal{V}(J_k)$ of J_k , which is a Zariski closed subset of the dual \mathfrak{h}^* of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . We deduce the “if” part of Main Theorem from a result of Frenkel and Malikov [FrM], which states that every G -integrable admissible representation at level k is a module over $L(k\Lambda_0)$, together with an affine analogue of the Duflo-Joseph Lemma (Lemma 2.3) for characteristic varieties. We prove the “only if” part of Main Theorem by reducing to the $\widehat{\mathfrak{sl}}_2$ -cases [AM] using the *semi-infinite restriction functor* $H^{\frac{\infty}{2}+i}(\mathfrak{m}_i[t, t^{-1}], ?)$ studied in [A3], where \mathfrak{m}_i is the nilradical of a minimal parabolic subalgebra of \mathfrak{g} , see §4.1 for the details.

It should be mentioned that there is another variety naturally associated with $A(L(k\Lambda_0))$, that is, the zero set $V(\text{gr } J_k)$ of the associated graded ideal $\text{gr } J_k$ of $\mathbb{C}[\mathfrak{g}^*]$. We have a surjection

$$(2) \quad R_{L(k\Lambda_0)} \twoheadrightarrow \text{gr } A(L(k\Lambda_0)) = \mathbb{C}[\mathfrak{g}^*] / \text{gr } J_k$$

of Poisson algebras, where $R_{L(k\Lambda_0)}$ is Zhu's C_2 -algebra of $L(k\Lambda_0)$, see [ALY, Proposition 3.3]. Hence $V(\text{gr } J_k)$ is contained in the *associated variety* [A1] $X_{L(k\Lambda_0)} = \text{Specm } R_{L(k\Lambda_0)}$. Although (2) is not an isomorphism in general, in a subsequent paper we prove the following.

Theorem 1.1. *Let k be an admissible number, with $k + h^\vee = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$. Then (2) induces the isomorphism of varieties*

$$V(\text{gr } J_k) \cong X_{L(k\Lambda_0)}.$$

Namely [A2]

$$V(\text{gr } J_k) \cong \begin{cases} \{x \in \mathfrak{g}; (\text{ad } x)^{2q} = 0\} & \text{if } (r^\vee, q) = 1, \\ \{x \in \mathfrak{g}; \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

which is an irreducible G -invariant subvariety of $\mathfrak{g}^* \cong \mathfrak{g}$. Here r^\vee is the lacing number of \mathfrak{g} , θ_s is the highest short root of \mathfrak{g} and π_{θ_s} is the irreducible finite-dimensional representation of \mathfrak{g} with highest weight θ_s .

Although they themselves are not rational in the usual sense² it has been conjectured [FKW, KW2] that admissible affine vertex algebras produce rational W -algebras in many cases by the method of the (generalized) quantum Drinfeld-Sokolov reduction. Exploiting the result in this article in a subsequent paper we prove the rationality of all the minimal series principal W -algebras [FKW].

²However Main Theorem implies that admissible affine vertex algebras are rational in the sense of [DLM].

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2. AFFINE VERTEX ALGEBRAS AND JOSEPH'S CHARACTERISTIC VARIETIES

Let \mathfrak{g} be a complex simple Lie algebra of rank l . Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

with a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . We will often identify \mathfrak{h} with \mathfrak{h}^* using the normalized invariant bilinear form $(\cdot | \cdot)$ of \mathfrak{g} . Let $\Delta, \Delta_+, \Delta_-, \Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of the roots, positive roots, negative roots, and simple roots of \mathfrak{g} , respectively. Also, let θ be the highest root of \mathfrak{g} , θ_s the highest short root, ρ the half sum of the positive roots, $W = \langle s_\alpha; \alpha \in \Delta \rangle \subset \text{Aut } \mathfrak{h}^*$ the Weyl group of \mathfrak{g} . Here $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, $\alpha^\vee = 2\alpha/(\alpha|\alpha)$. We set $s_i = s_{\alpha_i}$ for $i = 1, \dots, l$.

Denote by $\mathcal{O}^\mathfrak{g}$ the Bernstein-Gelfand-Gelfand category \mathcal{O} of \mathfrak{g} , by $L_\lambda^\mathfrak{g}$ the irreducible highest weight representation of \mathfrak{g} with highest weight $\lambda \in \mathfrak{h}^*$.

Let

$$\Upsilon : U(\mathfrak{g})_0 \rightarrow U(\mathfrak{h})$$

be the restriction of the projection $\Upsilon : U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}_+) \rightarrow U(\mathfrak{h})$ to $U(\mathfrak{g})_0 := \{u \in U(\mathfrak{g}); [h, u] = 0 \text{ for all } h \in \mathfrak{h}\}$. One knows that Υ is an algebra homomorphism.

For a two-sided ideal I of $U(\mathfrak{g})$, the *characteristic variety* [Jos] (without the ρ -shift) of I is defined as

$$\mathcal{V}(I) = \{\lambda \in \mathfrak{h}^*; p(\lambda) = 0 \text{ for all } p \in \Upsilon(I_0)\},$$

where $I_0 = I \cap U(\mathfrak{g})_0$. We have

$$(3) \quad \lambda \in \mathcal{V}(I) \iff IL_\lambda^\mathfrak{g} = 0.$$

Therefore $\mathcal{V}(I)$ classifies simple $U(\mathfrak{g})/I$ -module in the category $\mathcal{O}^\mathfrak{g}$.

The following fact is useful for us.

Lemma 2.1 ([Duf], [Jos, Lemma 2]). *Let I be a two-sided ideal of $U(\mathfrak{g})$, $\lambda \in \mathcal{V}(I)$. Suppose that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{N}$ for $\alpha_i \in \Pi$. Then $s_i \circ \lambda \in \mathcal{V}(I)$.*

Let

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

be the affine Kac-Moody algebra associated with \mathfrak{g} as in Introduction. The commutation relation of $\widehat{\mathfrak{g}}$ is given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m(x|y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = 0.$$

Let

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$$

be the triangular decomposition of $\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus \mathfrak{g}[t^{-1}]t^{-1}$, $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$, $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus \mathfrak{g}[t]t$. Let $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$ be the dual of $\widehat{\mathfrak{h}}$, where $\Lambda_0(K) = 1$, $\Lambda_0(\mathfrak{h}) = 0$. For $\lambda \in \widehat{\mathfrak{h}}^*$, denote by $\bar{\lambda} \in \mathfrak{h}^*$ the restriction of λ to \mathfrak{h} .

Let $\widehat{\Delta}^{re}$ be the set of real roots $\widehat{\mathfrak{g}}$ in the dual $\widehat{\mathfrak{h}}^* = \widehat{\mathfrak{h}}^* \oplus \mathbb{C}\delta$ of the extended Cartan subalgebra $\widehat{\mathfrak{h}}$, $\widehat{\Delta}^{re}$ the set of positive real roots, $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ the set of simple roots of $\widehat{\mathfrak{g}}$, where $\alpha_0 = -\theta + \delta$.

Let $\widehat{W} = W \ltimes Q^\vee$, the Weyl group of $\widehat{\mathfrak{g}}$, which is generated by s_0, s_1, \dots, s_l . Here $Q^\vee = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha^\vee$, the coroot lattice of \mathfrak{g} , and s_0 is the reflection corresponding to α_0 . Let $\widetilde{W} = W \ltimes P^\vee$, the extended Weyl group of $\widehat{\mathfrak{g}}$, where P^\vee is the coweight lattice of \mathfrak{g} . We have

$$\widetilde{W} = \widetilde{W}_+ \ltimes \widehat{W},$$

where \widetilde{W}_+ is the subgroup of \widetilde{W} consisting of elements which fix the set $\widehat{\Pi}$.

For $k \in \mathbb{C}$ let $U_k(\widehat{\mathfrak{g}})$ be the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ by the ideal generated by $K - k \text{ id}$. Denote by $U_k(\widehat{\mathfrak{g}})_\Delta$ the subspace of $U_k(\widehat{\mathfrak{g}})$ spanned by elements $(x_1 t^{n_1}) \dots (x_r t^{n_r})$ with $\sum_i n_i = -\Delta$, $x_i \in \widehat{\mathfrak{g}}$. This gives a \mathbb{Z} -grading on $U_k(\widehat{\mathfrak{g}})$: $U_k(\widehat{\mathfrak{g}}) = \bigoplus_{\Delta \in \mathbb{Z}} U_k(\widehat{\mathfrak{g}})_\Delta$. Define

$$\mathcal{U}_k(\widehat{\mathfrak{g}}) = \bigoplus_{\Delta \in \mathbb{Z}} \mathcal{U}_k(\widehat{\mathfrak{g}})_\Delta, \quad \mathcal{U}_k(\widehat{\mathfrak{g}}) = \varprojlim_N U_k(\widehat{\mathfrak{g}}) / \sum_{r > N} U_k(\widehat{\mathfrak{g}})_{r-\Delta} U_k(\widehat{\mathfrak{g}})_{-\Delta}.$$

Then $\mathcal{U}_k(\widehat{\mathfrak{g}})$ is a compatible degreewise complete topological algebra in the sense of [MNT]. Let $\mathcal{U}_k(\widehat{\mathfrak{g}})^{fin}$ be the dense subalgebra of $\mathcal{U}_k(\widehat{\mathfrak{g}})$ consisting of $\text{ad } \widehat{\mathfrak{g}}$ -finite vectors.

For $k \in \mathbb{C}$, let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with \mathfrak{g} at level k as in Introduction. By definition,

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$$

as a $\widehat{\mathfrak{g}}$ -module, where \mathbb{C}_k is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as a multiplication by k . The space $V^k(\mathfrak{g})$ is equipped with the natural vertex algebra structure (see [Kac, FBZ]). One knows [FZ] that there is a natural isomorphism

$$(4) \quad \mathcal{U}(V^k(\mathfrak{g})) \cong \mathcal{U}_k(\widehat{\mathfrak{g}})$$

of compatible degreewise complete topological algebras, where $\mathcal{U}(V)$ denotes the the current algebra [FZ] of V . Since

$$(5) \quad A(V) \cong \mathcal{U}(V)_0 / \overline{\sum_{\Delta > 0} \mathcal{U}(V)_\Delta \mathcal{U}(V)_{-\Delta}}$$

([FZ, NT]), where \overline{M} denotes the closure of M , (4) induces an algebra isomorphism

$$A(V^k(\mathfrak{g})) \cong U(\mathfrak{g}).$$

Here $A(V)$ denotes Zhu's algebra for a graded vertex algebra V as in Introduction. We will identify $A(V^k(\mathfrak{g}))$ with $U(\mathfrak{g})$ using this isomorphism. Also, it follows from (4) that a $V^k(\mathfrak{g})$ -module is the same as a smooth $\widehat{\mathfrak{g}}$ -module at level k . Here a $\widehat{\mathfrak{g}}$ -module M is called smooth if $(xt^n)m = 0$ for any $x \in \widehat{\mathfrak{g}}$, $m \in M$ and a sufficiently large n .

We assume that the level k be non-critical, that is, $k \neq -h^\vee$, where h^\vee is the dual Coxeter number of \mathfrak{g} . The vertex algebra $V^k(\mathfrak{g})$ is conformal by the Sugawara construction; let $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = Y(\omega, z)$, where $\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} x_i t^{-1} x_i t^{-1} \mathbf{1}$, $\{x_i\}$ is a basis of \mathfrak{g} , $\{x^i\}$ is the dual basis, and $\mathbf{1} = 1 \otimes 1 \in V^k(\mathfrak{g})$.

Let \mathcal{O}_k be the full subcategory of $\widehat{\mathfrak{g}}$ -modules consisting of objects M on which (1) K acts as the multiplication by k , (2) L_0 acts locally finitely, (3) $\widehat{\mathfrak{n}}_+$ acts locally nilpotently, (4) $\widehat{\mathfrak{h}}$ acts semisimply. Then \mathcal{O}_k can be considered as a full subcategory of the category of $V^k(\mathfrak{g})$ -modules. We regard an object $M \in \mathcal{O}_k$ a semisimple $\widehat{\mathfrak{h}}$ -module by letting D act on M as the semisimplification of the operator $-L_0$. Let $L(\lambda)$ be the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight λ .

Let N_k be the maximal ideal of $V^k(\mathfrak{g})$. The simple quotient

$$L(k\Lambda_0) := V^k(\mathfrak{g})/N_k$$

is called the (simple) affine vertex algebra associated with \mathfrak{g} at level k . A $V^k(\mathfrak{g})$ -module M is a $L(k\Lambda_0)$ -module if and only if the field $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ annihilates M for any $a \in N_k$.

For a vertex algebra V let $L(V)$ denote Borchard's Lie algebra associated with V . Then the current algebra $\mathcal{U}(V)$ of V is a quotient of a degreewise completion of the enveloping algebra $U(L(V))$ of $L(V)$, see [MNT] for the details. As N_k is an ideal of $V^k(\mathfrak{g})$ the image $L(N_k)$ of N_k in $L(V^k(\mathfrak{g}))$ is an ideal. Let $\mathcal{J}_k = \bigoplus_{\Delta} (\mathcal{J}_k)_{\Delta}$ be the degreewise closure of the ideal of $\mathcal{U}_k(\widehat{\mathfrak{g}})$ generated by the image of $L(N_k)$. Then

$$\mathcal{U}(L(k\Lambda_0)) \cong \mathcal{U}_k(\widehat{\mathfrak{g}})/\mathcal{J}_k, \quad \mathcal{U}(L(k\Lambda_0))_{\Delta} \cong \mathcal{U}_k(\widehat{\mathfrak{g}})_{\Delta}/(\mathcal{J}_k)_{\Delta}.$$

Set $\mathcal{J}_k^{\text{fin}} = \mathcal{J}_k \cap \mathcal{U}(\widehat{\mathfrak{g}})^{\text{fin}}$, $(\mathcal{J}_k^{\text{fin}})_{\Delta} = \mathcal{J}_k^{\text{fin}} \cap \mathcal{U}_k(\widehat{\mathfrak{g}})_{\Delta}$. Because \mathcal{J}_k is topologically generated by the image of $L(N_k)$ which consists of $\text{ad } \widehat{\mathfrak{g}}$ -finite vectors we have

$$(6) \quad \mathcal{J}_k M = 0 \iff \mathcal{J}_k^{\text{fin}} M = 0$$

for a $V^k(\mathfrak{g})$ -module M . Let J_k be the image of $(\mathcal{J}_k^{\text{fin}})_0$ under the projection $\mathcal{U}_k(\widehat{\mathfrak{g}})_0 \rightarrow U(\mathfrak{g})$. Then we have

$$(7) \quad A(L(k\Lambda_0)) \cong U(\mathfrak{g})/J_k.$$

The following is a direct consequence of (3) and Zhu's theorem [Zhu].

Proposition 2.2. *For $\lambda \in \widehat{\mathfrak{h}}^*$, $L(\lambda)$ is a module over $L(k\Lambda_0)$ if and only if $\bar{\lambda} \in \mathcal{V}(J_k)$.*

For a subspace M of $\mathcal{U}_k(\widehat{\mathfrak{g}})$, let \overline{M} denote the the degreewise closure of M . We have

$$(8) \quad \mathcal{U}_k(\widehat{\mathfrak{g}}) = U_k(\mathfrak{h}) \oplus \overline{(\widehat{\mathfrak{n}}_- \mathcal{U}_k(\widehat{\mathfrak{g}}) + \mathcal{U}_k(\widehat{\mathfrak{g}}) \widehat{\mathfrak{n}}_+)},$$

Let

$$\widehat{\Upsilon} : \mathcal{U}_k(\widehat{\mathfrak{g}}) \rightarrow U(\mathfrak{h})$$

be the projection with respect to the decomposition (8). Then we have

$$(9) \quad \Upsilon(J_k) = \widehat{\Upsilon}(\mathcal{J}_k^{\text{fin}}).$$

Lemma 2.3. *Let $\lambda \in \widehat{\mathfrak{h}}_k^*$ be a weight satisfying $\bar{\lambda} \in \mathcal{V}(J_k)$. Suppose that $\langle \lambda + \widehat{\rho}, \alpha_i^\vee \rangle \notin \mathbb{N}$ for $\alpha_i \in \widehat{\Pi}$. Then $\overline{s_i \circ \lambda} \in \mathcal{V}(J_k)$.*

Proof. The case $i = 1, \dots, l$ is the statement of Lemma 2.1. So let $i = 0$.

Let f_0 be a root vector of $\widehat{\mathfrak{g}}$ of root α_0 and set $\widehat{\mathfrak{b}}^{(i)} = \mathbb{C}f_0 \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+ = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{m}}$, the minimal parabolic subalgebra of $\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{l}}$ is its Levi subalgebra and $\widehat{\mathfrak{m}}$ its nilradical. Denote by $\widehat{\mathfrak{m}}_-$ the opposite subalgebra of $\widehat{\mathfrak{m}}$ so that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{m}}_- \oplus \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{m}}_+$. We have

$$\mathcal{U}_k(\widehat{\mathfrak{g}}) = \overline{U_k(\widehat{\mathfrak{l}})} \oplus \overline{(\widehat{\mathfrak{m}}_- \mathcal{U}_k(\widehat{\mathfrak{g}}) \oplus \mathcal{U}_k(\widehat{\mathfrak{g}}) \widehat{\mathfrak{m}}_+)},$$

where $U_k(\widehat{\mathfrak{l}}) = U(\widehat{\mathfrak{l}})/(K - k)U(\widehat{\mathfrak{l}})$.

Let $\widehat{\Upsilon}^{(0)} : \mathcal{U}_k(\widehat{\mathfrak{g}}) \rightarrow \overline{U_k(\widehat{\mathfrak{l}})}$ be the projection with respect to this decomposition. Note that $U_k(\widehat{\mathfrak{l}}) \subset \overline{U_k(\widehat{\mathfrak{l}})}$ coincides with the subalgebra of $\overline{U_k(\widehat{\mathfrak{l}})}$ consisting of the $\text{ad } \widehat{\mathfrak{l}}$ -finite vectors. Hence $\widehat{\Upsilon}^{(0)}(\mathcal{U}_k(\widehat{\mathfrak{g}})^{\text{fin}}) = U_k(\widehat{\mathfrak{l}})$, and

$$I := \widehat{\Upsilon}^{(0)}(\mathcal{J}_k^{\text{fin}})$$

is a two-sided ideal of $U_k(\widehat{\mathfrak{l}})$. By (9),

$$\Upsilon(J_k) = \gamma^{(0)}(I),$$

where $\gamma^{(0)} : U(\widehat{\mathfrak{l}}) \rightarrow U(\mathfrak{h})$ is the projection defined by the decomposition $U_k(\widehat{\mathfrak{l}}) = U(\mathfrak{h}) \oplus (f_0 U(\widehat{\mathfrak{l}}_0) + U(\widehat{\mathfrak{l}})e_0)$.

Now one can apply the argument of [Jos, Lemma 2] to obtain the assertion. \square

Lemma 2.4. *Let $\lambda \in \widehat{\mathfrak{h}}^*$ be a weight satisfying $\bar{\lambda} \in \mathcal{V}(J_k)$, and let $w \in \widetilde{W}$. Suppose that $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \widehat{\Delta}_+^{re} \cap w^{-1}(\widehat{\Delta}_-^{re})$. Then $\overline{w \circ \lambda} \in \mathcal{V}(J_k)$.*

Proof. The assertion follows from Lemma 2.3 in the case that $w \in \widehat{W}$. But one knows from [Li, §3] that if $L(\lambda)$ is a $L(k\Lambda_0)$ -module then so is $L(\pi \circ \lambda)$ for $\pi \in \widetilde{W}_+$. This completes the proof. \square

3. KAC-WAKIMOTO ADMISSIBLE REPRESENTATIONS

For $\lambda \in \widehat{\mathfrak{h}}^*$, let $\widehat{\Delta}(\lambda)$ and $\widehat{W}(\lambda)$ be its integral root system and its integral Weyl group, respectively.

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{re}; \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}, \quad \widehat{W}(\lambda) = \langle s_\alpha; \alpha \in \widehat{\Delta}(\lambda) \rangle.$$

Let $\widehat{\Delta}(\lambda)_+ = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}_+^{re}$, the set of positive root of $\widehat{\Delta}(\lambda)$ and $\Pi(\lambda) \subset \widehat{\Delta}(\lambda)_+$, the set of simple roots.

A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called admissible if

- (i) λ is regular dominant, that is, $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \{0, -1, -2, \dots\}$,
- (ii) $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$.

An admissible number (for $\widehat{\mathfrak{g}}$) is a complex number k such that $k\Lambda_0$ is admissible.

Proposition 3.1 ([KW1, KW2]). *A complex number k is admissible if and only if*

$$k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, (p, q) = 1, p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1 \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where h is the Coxeter number of \mathfrak{g} and r^\vee is the lacing number of \mathfrak{g} , that is, the maximal number of the edges in the Dynkin diagram of \mathfrak{g} . If this is the case we

have $\widehat{\Pi}(k\Lambda_0) = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_l\}$, where

$$\alpha_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1 \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

For an admissible number k let Pr_k be the set of admissible weights λ such that $\widehat{\Delta}(\lambda) \cong \widehat{\Delta}(k\Lambda_0)$. Set

$$Pr_k^+ = Pr_k \cap \widehat{P}_k,$$

where

$$\widehat{P}_k = \{\lambda \in \widehat{\mathfrak{h}}^*; \lambda(K) = k, \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i = 1, \dots, l\}.$$

Then [KW1]

(10)

$$Pr_k^+ = \begin{cases} \{\lambda \in \widehat{P}_k^+; \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for } i = 1, \dots, l, \langle \lambda, \theta \rangle \leq p - h^\vee\} & \text{if } (r^\vee, q) = 1, \\ \{\lambda \in \widehat{P}_k^+; \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for } i = 1, \dots, l, \langle \lambda, \theta_s^\vee \rangle \leq p - h\} & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

and we have

$$(11) \quad Pr_k = \bigcup_{\substack{y \in \widehat{W} \\ y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} := y \circ Pr_k^+.$$

Recall the following important result [FrM].

Theorem 3.2 ([FrM]). *Let k be an admissible number and $\lambda \in Pr_k^+$. Then $L(\lambda)$ is a module over $L(k\Lambda_0)$.*

Proof of the “if part” of Main Theorem. Let $\lambda \in Pr_{k,y}$. The condition $y(\widehat{\Delta}(k\Lambda_0)) \subset \widehat{\Delta}_+^{re}$ implies that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \widehat{\Delta}_+^{re} \cap y^{-1}(\widehat{\Delta}_-^{re})$. Hence the assertion follows from Proposition 2.2, Lemma 2.4 and Theorem 3.2. \square

4. SEMI-INFINITE RESTRICTION FUNCTORS

Let $\{e_i, h_i, f_i; i = 1, \dots, l\}$ be a set of Chevalley generators of \mathfrak{g} . For $i = 1, \dots, l$, let \mathfrak{p}_i be the minimal parabolic subalgebra $\mathbb{C}f_i \oplus \mathfrak{h} \oplus \mathfrak{n}$, \mathfrak{l}_i its Levi subalgebra, \mathfrak{m}_i its nil-radical. We have

$$\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{m}_i, \quad \mathfrak{l}_i = \mathfrak{sl}_2^{(i)} + \mathfrak{h},$$

where $\mathfrak{sl}_2^{(i)}$ is the copy of \mathfrak{sl}_2 spanned by e_i, h_i and f_i .

Let

$$L\mathfrak{m}_i = \mathfrak{m}[t, t^{-1}] \subset \widehat{\mathfrak{g}},$$

$$\widehat{\mathfrak{sl}}_2^{(i)} = \mathfrak{sl}_2^{(i)}[t, t^{-1}] \oplus \mathbb{C}K_i \subset \widehat{\mathfrak{g}}, \quad \text{where } K_i = \frac{2}{(\alpha_i | \alpha_i)} K.$$

For an object M of \mathcal{O}_k we shall consider the semi-infinite $L\mathfrak{m}_i$ -cohomology $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)$ with coefficient in M . This is defined by the Feigin's standard complex $(C^\bullet(L\mathfrak{m}_i, M), d)$ [Fei]: $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M) = H^i(C^\bullet(L\mathfrak{m}_i, M), d)$. The extended Cartan subalgebra $\widehat{\mathfrak{h}}$ acts naturally on the space $C^\bullet(L\mathfrak{m}_i, M)$ and we have the weight space decomposition $C^\bullet(L\mathfrak{m}_i, M) = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*} C^\bullet(L\mathfrak{m}_i, M)_\lambda$, where $C^\bullet(L\mathfrak{m}_i, M)_\lambda$ is

the weight space of weight λ . Because the action of $\tilde{\mathfrak{h}}$ commutes with that of d we have

$$H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M) = \bigoplus_{\lambda \in \tilde{\mathfrak{h}}^*} H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_\lambda,$$

where $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_\lambda$ is the cohomology of the subcomplex $C^\bullet(L\mathfrak{m}_i, M)_\lambda$.

If M is an object M of \mathcal{O}_k of $\hat{\mathfrak{g}}$ then $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}^{(i)}, M)$, $p \in \mathbb{Z}$, is naturally a module over $\hat{\mathfrak{sl}}_2^{(i)}$ on which K_i acts as a multiplication by k_i , where

$$k_i + 2 = \frac{2}{(\alpha_i | \alpha_i)}(k + h^\vee),$$

see e.g. [HT]. In particular if $k + h^\vee = p/q$ then

$$(12) \quad k_i + 2 = \begin{cases} \frac{p}{q} & \text{if } \alpha_i \text{ is a long root,} \\ \frac{r_\vee p}{q} & \text{if } \alpha_i \text{ is a short root.} \end{cases}$$

Therefore if k is an admissible number for $\hat{\mathfrak{g}}$, then k_i is an admissible number for $\hat{\mathfrak{sl}}_2$.

Let $\hat{\mathfrak{h}}_i = \mathbb{C}h_i \oplus \mathbb{C}K_i$, the Cartan subalgebra of $\hat{\mathfrak{sl}}_2^{(i)}$. For $\lambda \in \hat{\mathfrak{h}}^*$, define $\lambda^{(i)} \in \hat{\mathfrak{h}}_i^*$ by

$$\lambda^{(i)}(h_i) = \lambda(h_i), \quad \lambda^{(i)}(K_i) = k_i$$

Then a vector in $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_\lambda$ has the weight $\lambda^{(i)}$ as a module over $\hat{\mathfrak{sl}}_2^{(i)}$.

For $M \in \mathcal{O}_k$ let

$$M_{[d]} = \{m \in M; Dm = L_0 m\} = \bigoplus_{\substack{\lambda \in \tilde{\mathfrak{h}}^* \\ \langle \lambda, D \rangle = d}} M_\lambda.$$

Note that each $M_{[d]}$ is a \mathfrak{g} -submodule of M . Similarly, set $C^\bullet(L\mathfrak{m}_i, M)_{[d]} = \{c \in C(L\mathfrak{m}_i, M); Dm = dm\}$. Then $C^\bullet(L\mathfrak{m}_i, M)_{[d]}$ is a subcomplex of $C^\bullet(L\mathfrak{m}_i, M)$ and we have

$$H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M) = \bigoplus_{d \in \mathbb{C}} H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_{[d]},$$

where $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_{[d]}$ is the cohomology of the subcomplex $C^\bullet(L\mathfrak{m}_i, M)_{[d]}$.

Lemma 4.1. *Let $M \in \mathcal{O}_k$ such that $M = \bigoplus_{d \leq d_0} M_{[d]}$. Then $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M) = \bigoplus_{d \leq d_0} H^{\frac{\infty}{2}+\bullet}(L\mathfrak{m}_i, M)_{[d]}$ and*

$$H^{\frac{\infty}{2}+p}(L\mathfrak{m}_i, M)_{[d_0]} \cong \begin{cases} H^p(\mathfrak{m}, M_{[d_0]}) & \text{if } p \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assertion follows by observing that $C^\bullet(L\mathfrak{m}_i, M)_{[d_0]} = M_{[d_0]} \otimes \bigwedge^\bullet(\mathfrak{m}_i)$ and the restriction of the differential to $C^\bullet(L\mathfrak{m}_i, M)_{[d_0]}$ coincides with the differential of the Chevalley complex for calculating the (usual) Lie algebra cohomology $H^\bullet(\mathfrak{m}_i, M_{[d_0]})$. \square

Theorem 4.2 ([A3]). *Let k be an admissible number. If M is a module over $L(k\Lambda_0)$ then $H^{\frac{\infty}{2}+p}(L\mathfrak{m}^{(i)}, M)$, $p \in \mathbb{Z}$, is a direct sum of irreducible admissible representations of $\hat{\mathfrak{sl}}_2^{(i)}$ of level k_i .*

Lemma 4.3. *Let k be an admissible number and suppose that $L(\lambda)$ is module over $L(k\Lambda_0)$. Then $\lambda^{(i)}$ is an admissible weight for $\widehat{\mathfrak{sl}}_2$ for all $i = 1, \dots, l$.*

Proof. By Lemma 4.1, $H^{\frac{\infty}{2}+0}(L\mathfrak{m}_i, L(\lambda)) = \bigoplus_{d \leq \lambda(D)} H^{\frac{\infty}{2}+0}(L\mathfrak{m}_i, L(\lambda))_{[d]}$ and

$$H^{\frac{\infty}{2}+0}(L\mathfrak{m}_i, L(\lambda))_{[\lambda(D)]} = H^{\frac{\infty}{2}+0}(L\mathfrak{m}_i, L(\lambda))_{\lambda} \cong \mathbb{C}.$$

Hence by Theorem 4.2 $H^{\frac{\infty}{2}+0}(L\mathfrak{m}_i, L(\lambda))_{\lambda}$ must generate an irreducible admissible representation of $\widehat{\mathfrak{sl}}_2^{(i)}$ with highest weight $\lambda^{(i)}$. This completes the proof. \square

Proposition 4.4. *Suppose that $L(\lambda)$ is a $L(k\Lambda_0)$ -module. Then $\widehat{\Delta}(\lambda) \cong \widehat{\Delta}(k\Lambda_0)$.*

Proof. By Lemma 4.3, $\langle \lambda + \rho, \alpha_i^{\vee} \rangle \in \frac{2}{(\alpha_i | \alpha_i)q} \mathbb{Z}$ for all $i = 1, \dots, l$. It follows that there exists $n_i \in \mathbb{Z}$ for each $i = 1, \dots, l$ such that $\alpha_i + n_i \delta \in \widehat{\Delta}(\lambda)$. Hence

$$(13) \quad \text{there exists } n_{\alpha} \in \mathbb{Z} \text{ for each } \alpha \in \Delta \text{ such that } \alpha + n_{\alpha} \delta \in \widehat{\Delta}(\lambda).$$

This implies that $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{re}$. Therefore [KW1] $\widehat{\Delta}(\lambda)$ is isomorphic to the integral root system of some admissible weight of level k . By the classification of admissible weights [KW1] and the property (13) it follows that $\widehat{\Delta}(\lambda)$ must be isomorphic to $\widehat{\Delta}(k\Lambda_0)$. \square

Proposition 4.5. *Let $\lambda \in \widehat{P}_k$ and suppose that $L(\lambda)$ is a module over $L(k\Lambda_0)$. Then $L(\lambda)$ is admissible.*

Proof. First, it follows that $\langle \lambda, \alpha_i^{\vee} \rangle \geq 0$ for $i = 1, \dots, l$ from Lemma 4.3. Therefore the \mathfrak{g} -submodule $L_{\lambda}^{\mathfrak{g}}$ generated by the highest weight vector of $L(\lambda)$ is finite-dimensional. By [Kos] we have

$$H^i(\mathfrak{m}_i, L_{\lambda}^{\mathfrak{g}}) \cong \bigoplus_{\substack{w \in W^{(i)} \\ \ell(w)=i}} L_{w \circ \bar{\lambda}}^{l_i}$$

as \mathfrak{l}_i -modules, where $L_{\bar{\mu}}^{l_i}$ is the irreducible highest weight representation of \mathfrak{l}_i with highest weight $\bar{\mu}$ and

$$W^{(i)} = \{w \in W; w^{-1}(\alpha_i) \in \Delta_+\}.$$

Therefore

$$H^{\frac{\infty}{2}+p}(L\mathfrak{m}_i, L(\lambda))_{[\lambda(D)]} \cong \bigoplus_{\substack{w \in W^{(i)} \\ \ell(w)=i}} L_{w \circ \bar{\lambda}}^{l_i}$$

by Lemma 4.1. In particular

$$(14) \quad (w \circ \lambda)^{(i)} \text{ must be an admissible weight for } \widehat{\mathfrak{sl}}_2 \text{ for all } w \in W^{(i)}, i = 1, \dots, l.$$

Suppose that $(r^{\vee}, q) = 1$. Let α_i be any simple long root. Then there exists $w \in W$ such that $\theta = w^{-1}(\alpha_i)$. By (10), (12) and (14),

$$p - 2 \geq \langle w \circ \lambda, \alpha_i^{\vee} \rangle = \langle w(\lambda + \rho), \alpha_i^{\vee} \rangle - 1 = \langle \lambda + \rho, \theta \rangle - 1 = \langle \lambda, \theta \rangle + h^{\vee} - 1 - 1.$$

Therefore $\langle \lambda, \theta \rangle \leq p - h^{\vee}$, and hence $\lambda \in Pr_k^+$ by (10).

Next suppose that $(r^{\vee}, q) = r^{\vee}$. Let α_i be any simple short root. Then there exists $w \in W$ such that $\theta_s = w^{-1}(\alpha_i)$. By (10), (12) and (14),

$$p - 2 \geq \langle w \circ \lambda, \alpha_i^{\vee} \rangle = \langle w(\lambda + \rho), \alpha_i^{\vee} \rangle - 1 = \langle \lambda + \rho, \theta_s^{\vee} \rangle - 1 = \langle \lambda, \theta_s^{\vee} \rangle + h - 1 - 1.$$

Therefore $\langle \lambda, \theta_s^\vee \rangle \leq p - h$, and hence $\lambda \in Pr_k^+$. \square

Proof of the “only if” part of Main Theorem. Suppose that $L(\lambda)$ is module over $L(k\Lambda_0)$. By Proposition 4.4, $\widehat{\Delta}(\lambda) \cong \widehat{\Delta}(k\Lambda_0)$. Therefore by [KW1] there exists an element $\mu \in \widehat{P}_k$ and $y \in \widehat{W}$ such that $\widehat{\Delta}(\mu) = \widehat{\Delta}(k\Lambda_0)$, $y(\widehat{\Delta}(\mu)) \subset \widehat{\Delta}_+^{re}$ and $\lambda = y \circ \mu$. It follows from Lemma 2.4 that $L(\mu)$ is also a module over $L(k\Lambda_0)$. Hence μ must be an admissible weight by Proposition 4.5, and therefore, so is λ . This completes the “only if” part of Main Theorem. \square

The conjecture of Adamović and Milas [AM, Conjecture 3.5.7] now follows immediately from the fact that $\text{Ext}_{\mathcal{O}_k}^1(L(\lambda), L(\mu)) = 0$ for any admissible weights λ, μ of level k since they are regular dominant weights. \square

Remark 4.6. In fact it is known by [GK] that $\text{Ext}_{\mathfrak{g}}^1(L(\lambda), L(\mu)) = 0$ for any (not necessarily distinct) admissible weights λ, μ .

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